ANALYZING STUDENTS' CONSTRUCTION OF THE RELATIONSHIP BETWEEN CONTINUITY AND DIFFERENTIABILITY

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Abstract:

The relationship between continuity and differentiability in graphical problems are two concepts that students are pivotal in understanding derivative concepts. However, researchers rarely pay attention to how students understand the relationship between Continuity and differentiability or vice versa. This study investigates students' understanding of the relationship between continuity and differentiability. This study is exploratory and focuses on the meanings students construct. The participants were 195 third-year undergraduate students from various Indonesian universities. A questionnaire and interview were used to collect data. Ten of the participants agreed to an in-depth interview for exploration and clarification. Thematic analysis was used to deduce patterns from participants' responses based on the findings. The results indicated that students construct three types of meanings when they solve problems: physical, analytical, and covariational. The findings could serve as a conceptual framework for future learning processes emphasizing continuity and differentiability.

Abstrak:

Hubungan antara kontinuitas dan diferensiasi dalam soal grafis merupakan dua konsep yang sangat penting bagi siswa dalam memahami konsep turunan. Namun peneliti jarang memperhatikan bagaimana siswa memahami hubungan antara Kontinuitas dan diferensiasi atau sebaliknya. Penelitian ini menyelidiki pengetahuan siswa tentang hubungan antara kontinuitas dan diferensiasi. Penelitian ini bersifat eksploratif dan berfokus pada makna yang dibangun siswa. Pesertanya adalah mahasiswa S1 tahun ketiga dari berbagai universitas di Indonesia yang berjumlah 195 orang. Kuesioner dan wawancara digunakan untuk mengumpulkan data. Sepuluh peserta menyetujui wawancara mendalam untuk eksplorasi dan klarifikasi. Analisis tematik digunakan untuk menyimpulkan pola tanggapan peserta berdasarkan temuan. Hasilnya menunjukkan bahwa siswa membangun tiga jenis makna ketika mereka memecahkan masalah: fisik, analitis, dan dan kovarian. Temuan ini dapat berfungsi sebagai kerangka konseptual untuk proses pembelajaran di masa depan yang menekankan kontinuitas dan diferensiasi.

Keywords:
Relationship, Continuity, Differentiability, Graphical Problems

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INTRODUCTION

Numerous researchers in mathematics education have taken an interest in calculus in recent years (Tall, 2008). Calculus is a fundamental area of mathematics that lies at the heart of the mathematics curriculum from middle school to university (Tsamir & Ovodenko, 2013). In our preliminary research, we discovered several studies examining students' difficulties comprehending calculus concepts due to their lack of understanding of the topic of function (Carlson, Jacobs, & Coe 2002; Zandieh & Knapp, 2006). Additionally, several studies have been conducted on students' conceptual and procedural understanding of the derivative and antiderivative relationships (García-García & Dolores-Flores, 2019; Rittle-Johnson, Schneider, & Star, 2015). Finally, some research examined the application of calculus concepts to real-world situations or graphs (García-García & Dolores-Flores, 2019; Ikram, Purwabto, & Parta, 2020; Jones, 2017). Calculus's curriculum places a greater emphasis on algebraic representation. As a result, students become accustomed to manipulating algebraic expressions rather than identifying concepts and definitions and performing theorem analysis (Huang, 2014). Nonetheless, studies examining the relationship between concepts and their various representations, such as the relationship between continuity and differentiability in graph problems, are few and far between. Numerous students frequently overlook the two relationships and attempt to avoid the situation's complications (Fuentealba, Sánchez-Matamoros, & Badilo, 2017). Additionally, several students graphically misunderstood the statement, "If f is differentiable at $x = a$, then f is continuous at $x = a$" (Siyepu, 2015). Thus, students should understand the relationship between continuity and differentiability because it will aid them in comprehending the calculus concept.

Although many students can successfully solve calculus problems for algebra cases, our literature review in the International Journal of Research in Mathematics Education revealed a lack of research on the relationship between continuity and differentiability in algebra. By examining the students, we can provide references for the classroom's calculus learning process, emphasizing the application of the relationship between continuity and differentiability. Teachers can minimize the use of algebra problems to instil calculus concepts (Dibbs, 2019). Numerous expressions involving continuity and differentiability require logic. Students who do not understand the logic behind a statement will have difficulty comprehending and interpreting the proof of a theorem (Mcgowen & Tall, 2013).

Additionally, students' inability to analyze the meaning of a statement's quantification resulted in several errors, such as "if-then," only if," if and if only," "$\in$, " and "$\forall$". For instance, in the calculus book, there are three statements, and they are: (1) differentiability implies continuity; (2) if f is differentiable at c, then f is continuous at c; and (3) if a function is not continuous at a point, then the function is not differentiable. Providing that the students think the other way around, they might realize that continuity does not ensure differentiability, which means that the statement's converse is incorrect (Sevimli, 2018a). Thus, they frequently employed counterexamples to demonstrate that an idea is false.

Several studies indicated some difficulties students encounter when applying the concept of continuity and differentiability. For example, learners found it difficult to elucidate why $f(x) = |\sin x|$ is not differentiable at $x = \pi$ by employing an analytic approach (Biza & Zachariades, 2010; Mcgowen & Tall, 2013). Additionally, students were less able to coordinate the connection between continuity and differentiability when sketching graphs, causing them to miss the relationship for each interval (Baker, Cooley, & Trigueros, 2000; Cooley, Trigueros, & Baker, 2007). It demonstrates that analytic rather than visual processes dominate individuals' thoughts (Haciomeroglu, Aspinwall, & Presmeg, 2010). While most students understood that a function is differentiable if the graph contains a tangent, they cannot describe the process by which the domain value is determined. Finally, they were unaware of the condition that prevents a function from being differentiable based on its graphic representation (de Almeida & da Silva, 2018). As a result, we can investigate the meanings students construct about the relationship between continuity and differentiability through graphic visualization.

In Indonesia's calculus curriculum, the derivative concept is applied to graphs. The majority of students, on the other hand, focused exclusively on the general properties of the function (for example, the shape of the curve or predicting the function's formula) rather than on the derivative's properties (continuity, going up or going down, and stationary). They rarely develop their ideas into solutions to graph-related problems. Thus, encouraging them to complete the derivative task will help them develop their skills. This study demonstrates the critical nature of providing students with multiple interpretations of mathematical concepts.

In summary, this study is relevant and necessary because (1) research on the relationship between continuity and differentiability in graphic problems is still uncommon; (2) the relationship between continuity and differentiability should be the primary focus for students in all countries, including Indonesia; and (3) this study provides information regarding the meanings that the students build as references to teach continuity and differentiability in the future research. This study is a continuation of the research conducted by Ikram, Purwanto, & Parta (2020) on the reasoning displayed by students when sketching graphs involving the derivative concept. The study hypothesized that discrepancies exist when students draw a graph with continuity and differentiability. As a result, the study concentrated on the issue of whether a function is continuous and differentiable at a point. It was then used to address the research question regarding the interpretations students make to obtain answers. Thus, students are likely to conduct various analyses to ascertain the correct answers involving the curve behaviour, where the function is decreasing or increasing, to solve the problem analytically.

Individuals are stimulated to construct a reverse process between the derivative graph and its antiderivative by connecting the function and its derivative. When students draw a curve, they encounter the standard procedure, which includes (1) determining the monotony (whether the curve is ascending or descending) in an interval; (2) determining the curve's highest and lowest points (the vertex); (3) determining the curve's turning point and extreme point; and (4) using the second derivative to determine the curve's concavity. However, the procedures do not improve students' conceptual understanding, such as the meaninglessness of the curve's concavity or whether the curve is going down or up (Berry & Nyman, 2003). Numerous studies examined students' comprehension of the relationship between a function and its derivative (Haciomeroglu, Aspinwall, & Presmeg, 2010; Hong & Thomas, 2015). Their findings indicated that learners encountered difficulties and misconceptions when attempting to interpret the relationship between the derivative graph and its antiderivative, including the extreme point, horizontal tangent at a point, and the symbol for the second derivative. The results provide a preliminary overview of how difficult it is for students to sketch the graph of a function.

The difference in students' preferences also affects how they interpret graphs (Haciomeroglu, Aspinwall, & Presmeg, 2010). Visual thinkers frequently determine the function's graph by observing the slope change at the derivative curve but fail to interpret the shift near the graph's vertical tangent based on the function derivative graph. Analytic thinkers are prone to relate problems to their algebraic expressions but have difficulty associating continuity and differentiability. Additionally, Hong & Thomas (2015) propose two distinct thinking models for children when they construct a graph, namely a process dominated by (1) algebraic thinkers and (2) interval thinkers. The distinction between the two models is determined by an awareness of the meanings and relationships between the elements in the problems encountered, the presence of knowledge reconstruction to create new structures, and the ability to rearrange existing knowledge (García-García & Dolores-Flores, 2019). It demonstrates the importance of synthesizing students' thought processes to supplement students' comprehension of calculus concepts.

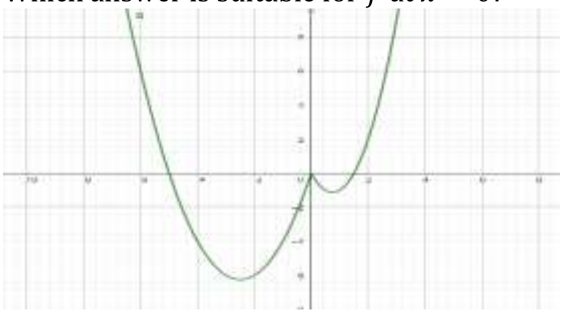
Continuity and differentiability are essential concepts in calculus that involve formal definition limits and various theorems (Swinyard & Larsen, 2012). However, the students seem to recognize the limit as a function at a point or as objects to solve problems formally (Fernández-Plaza & Simpson, 2016). Thus, the connection between continuity and differentiability is less coherent for them. Some textbooks have outlined theorems related to the concepts, for example: (1) a function is continuous at a point if $\lim_{x\to c} f(x)$ and $f(c)$ (c is an element of the domain of f) exist, and $\lim_{x\to c} f(x) = f(c)$; (2) f is differentiable at c if $\lim_{h\to 0}\frac{f(x+c)-f(x)}{h}$ exist; and (3) if f is differentiable at c, then f is continuous at c (for every function $f: \mathbb{R} \to \mathbb{R}$ and $c \in \mathbb{R}$). On the other hand, if a function is discontinuous at a point (as a result of a jump, for example), it is not differentiable at that point (Sevimli, 2018b). There is a case in which a function is continuous but not differentiable (for example, the vertical tangent or when the function has a high degree of curve). Students should analyze situations in which continuity does not necessarily imply differentiability.

RESEARCH METHOD

The study is exploratory in nature and focuses on the meanings students construct when they apply the relationship between continuity and differentiability to graph problems. The study recruited 195 third-year undergraduate students from a variety of universities. Their ages ranged between 19 and 20 years. The students had 45 minutes to complete the problems in Google Forms. We contacted numerous colleagues at numerous colleges to ensure their students successfully completed the assigned task. All students participating in the study were enrolled in an advanced calculus course. According to their universities' calculus curricula, most students were taught how to solve problems procedurally (For instance, determining the derivative of a function and drawing a graph with the known function formula). As a result, they had few opportunities to apply calculus concepts to graph problems. Numerous students struggled with analyzing the relationship between continuity and differentiability. Additionally, some students analyzed continuity alone, disregarding differentiability. The results section discusses the students' responses.

A questionnaire and an interview were used to collect data. The questionnaire included three tasks, each containing references to continuity and differentiability. The assignment is as follows.

Table 1. Task Development

Task	Description
Task 1 The following figure is the graph of $f(x)$. Which answer is suitable for f at $x = 0$?  ○ f is continuous but is not differentiable ○ f is discontinuous but is differentiable ○ f is continuous and differentiable ○ f is discontinuous and not differentiable	The graph of f given has a kink at $x = 0$. The results expected are: 1. Students interpret the curve behaviour of f near $x = 0$; for example, f is increasing if $x < 0$ and is decreasing immediately if $x > 0$. 2. Students use the limit definition to obtain the conclusion about continuity and differentiability. 3. Students apply the relation between continuity and differentiability, but they must consider the sharp corner of f at $x = 0$. 4. Students construct their ideas by employing various representations to find solutions.

We formulated the task with some considerations. First, graph aspects of concepts of continuity and differentiability are critical for students' conceptual understanding and perspective. Second, we discovered that most students concentrated exclusively on problems involving algebraic representations. One involving graphical representations was rarely provided. It reveals that students' experience and the inability to apply the algorithm to obtain solutions led us to believe that students' thinking was challenged in interpreting the graph of f and applying concepts of continuity and differentiability. Finally, each task includes graphical representations. Ten students participated in a follow-up interview to clarify their responses based on the solution obtained. We conducted interviews with students who provided an interesting response to the questionnaire. We showed their original solutions during the interview. They were tasked with providing detailed justifications for the concepts they wrote. The interview lasted approximately 10-15 minutes and was recorded and transcribed using an audio recorder.

We analyzed the data using thematic analysis. It deduces patterns (themes) from respondents' responses using the instruments provided (Braun & Clarke, 2006). We chose this method for a variety of reasons. Firstly, there is no prior framework for examining the meanings of continuity and differentiability, which could contribute to this study. Secondly, the method is adaptable and could be used to address the research question, specifically regarding the meanings students construct when applying the relationship between continuity and differentiability to the graph problem. We can then use the methods to analyze a large data set.

We discovered patterns as well as unique naming using the thematic analysis method. The contribution of this study is significant through thematic analysis, as studies utilizing the method in mathematics education are still uncommon. To accomplish our objectives, we adopted the phases proposed by Braun & Clarke (2006).

There were six stages. The first phase was to familiarize ourselves with the data. We read students' work and interview transcripts during this phase. It assisted us in identifying the concepts, verbal expressions, and thought processes revealed during the interview. The second phase was generating the initial code. We developed preliminary codes based on our general reading of the transcript and the conceptual framework used. To illustrate the relationship between continuity and differentiability, we examined participants' verbal expressions while solving problems, such as the following interview result:

Interviewer : What is your answer in Task 1?
Student 2 : f is continuous at $x = 0$, but it does not have a derivative.
Interviewer : How do you come to that conclusion?
Student 2 : The graph seemed continuous, without holes, jumps, and distance. Thus, I conclude that f is continuous at $x = 0$. However, the curve also has a high degree of curve at $x = 0$. It means that f has no derivative at the point or is not differentiable at $x = 0$.
Interviewer : What is the solution to Task 1?
Student 3 : f is continuous, but it is not differentiable?
Interviewer : What do you think?
Student 3 : f is continuous at $x = 0$ because the value of limit as x approach

0 exists and equals $f(0)$. However, it is not differentiable at $x = 0$ because the value of its derivative or

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0}$$

The values from the right and the left are different. Thus, I utilize the limit concept to conclude.

We generated codes based on the bolded verbal expressions. For instance, Student 2 employed curve behaviour to solve problems, whereas Student 3 relied on analytical identification. We created eight codes in this phase to represent the meanings of the relationship between continuity and differentiability.

Furthermore, the third phase was looking for themes. In this phase, we made, determined, and modified codes to understand relationships and formed themes. We grouped codes with the same meanings; for example, the code "f curve is continuous and has a sharp corner at $x = 0$", and the code "f curve has a kink at $x = 0$". In this case, we utilized the theme of physical meaning because the codes' main characteristics showed that students concluded by using the behaviour around the point. There was also the theme of analytical meaning. We grouped the code "The $f(0)$ as well as the right-hand and left-hand limits of $f(x)$ as x approach zero exists, but

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0}$$

Have different values when it approaches zero from the left and the right" and "continuous because the limit value exists, but is not differentiable because its derivative values are different." It implied that students employ analytic properties to investigate the problems. We also used the graphical meaning to form other codes. For example, the code "derivative is the gradient of the curve tangent," or "the curve f is increasing as x approach 0 from the left and the right, and its derivative value decreases, decreasing". We need to consider it as visual meaning because they use changes near x to make conclusions.

The fourth phase was reviewing themes. We repeatedly discussed the relationship between the themes and our data during this phase. The process would become a stage of obtaining the relationship between continuity and differentiability students construct. The code was regarded as having physical meaning in the case of covariational meaning. Nonetheless, our team discussions concluded that we must develop a new theme. We believe it has its answer patterns through curve behaviour and visual representations of derivatives.

The fifth phase was defining and naming themes. We defined and labelled the three ways in which students develop meanings for the connection between continuity and differentiability: physical meaning, analytical meaning, and visual meaning.

The last phase was producing the report. In this phase, we wrote the final reports of our research results. Besides, we carried out data triangulation to enhance the objectivity of our findings. We improved trustworthiness by discussing our research results with experts in mathematics education to achieve mutual agreements. We ensured that the data obtained was accurate and complete by administering the task in written form and transcribing every interview immediately after recording it. There was also a validation of the coding process and recoding of different categories through discussion with several mathematics education experts.

RESULTS AND DISCUSSION

In this section, we present our results from 195 students participating in this study. We specifically highlight students' responses in Task 1, where 156 out of 195 students answered that f is continuous and differentiable at x = 0. We collected several answers about why they chose the solutions, and Table 2 presents the summary.

Table 2. Students' Answers in Task 1

Answer	Responses	Number of Students
f is continuous and differentiable at $x = 0$	Meet the conditions of continuity and differentiability.	32
	Continuous because both of the limits as x approach 0 are 0. It is differentiable due to $f'(x) > 0$ from the left of $x = 0$ and $f'(x) < 0$ from the right of $x = 0$.	64
	The graph is continuous, which means that f has a derivative at $x = 0$.	60
f is continuous but is not differentiable at $x = 0$	Ignoring the differentiability's properties.	19
	The curve f is continuous but has a sharp corner at a point.	8
	Continuous because $\lim_{x \to 0} f(x)$ exists. It is not differentiable because $f'(0) = \lim_{x \to 0} \frac{f(x)-f(0)}{x-0}$ did not exist.	7
	The gradient of f at $x = 0$ is different.	5

Students' data who provide incorrect responses in this study become the participants of subsequent research. Following that, 39 students out of 156 responded that f is continuous but not differentiable at x = 0. They cited a variety of reasons. This section is the primary focus of the investigation to determine the answer to the research question. Nineteen students tend to ignore the properties of differentiability to solve the problems. They reasoned that the graph of f is connected between the intervals x > 0 and x < 0. Additionally, they stated that the curve near x = 0 lacks jumps and holes. This indicates that they overlooked the differentiability properties in order to reach conclusions. Additionally, we interviewed them to clarify their responses as follows.

Interviewer : Why did you answer that f is continuous but is not differentiable at $x = 0$?

Student 1 : The graph of f at $x = 0$ has no interference, for example, no hole, jump, or cuts.

Interviewer : How about the condition of its differentiability?

Student 1 : If it does not have holes or jumps, it is differentiable at x=0 or when f is continuous, but I do not know why it does not have a derivative.

The result shows that students overlooked the properties of differentiability because they believe continuity results in differentiability. In other words, they solved the problem by applying procedural knowledge.

Furthermore, eight out of forty-three students correctly stated that f is continuous but not differentiable at $x = 0$. The reason was the curve behaviour of f, which is continuous and forms breaks. It shows that they recognized that if the curve of f has a break or kink, it does not have a derivative value. Yaqin, one of the participants, answered the question by utilizing the gradient of the curve tangent and predicting the graph's function formula. He stated that (1) the slope of f is different from the left and the right of $x = 0$ (1 and -1), indicating that the function does not have a derivative at $x = 0$; and (2) the graph of f contains two curves, indicating that it is a piecewise function. At $x = 0$, the function has breaks, indicating no value. We interviewed two students to ascertain the specific causes of these difficulties.

Interviewer : Why did you say that f is continuous but is not differentiable at $x = 0$?

Student 2 : The graph seems continuous without hole, jump, or distance, so I conclude that f is continuous at $x = 0$. However, it has a sharp corner at $x = 0$. It implied that f does not have a derivative at the point or is not differentiable at $x = 0$.

Students concentrated on the curve behaviour of f to conclude its differentiability and continuity. This statement indicated that they interpreted the situation as having a physical meaning. They reasoned that the continuous curve demonstrates continuity properties (the curve of f which does not have a jump, hole, or asymptote). A sharp corner in the curve demonstrates the properties of differentiability. We believe their observation of curve behaviours was critical and warranted further discussion in this study. Students' constructed meanings provide valuable insights and references for the classroom teaching process.

Apart from using the curve behaviour of f to answer the question, 7 out of 39 participants used the analytical properties of continuity and differentiability by employing formal definitions. For instance, Ristia explained that the graph of f is continuous because it does not have a jump. She also added that its curve meets the conditions of continuity in function, that is, (1) the right-hand and left-hand are defined or $\lim_{x\to 0} f(x)$ exists; (2) the function value at $x = 0$ exists; and (3) the value of its limit and its function is the same or $\lim_{x\to 0} f(x) = f(0)$. Next, to identify its differentiability, the participant used the definition of derivative, that is,

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0}$$

(examining when $x \to 0^-$ and $x \to 0^+$). Moreover, the participant also employs the ideas of the previous problem (e.g., $f(x) = |x|$) to conclude. It implied that students used analytical meaning to explain the relationship between continuity and differentiability.

We deem this necessary to be explored as a reference in the learning process. The results of our interviews with the participants who fall into this category are as follows.

Interviewer : How did you realize that f is continuous but is not differentiable at $x = 0$?

Student 3 : Based on the function graph, it is continuous because it does not have a jump. Formally, it met the conditions of continuity of a function; they are the limit that exists (the right-hand and the left-hand limit are the same), and the function value at $x = 0$ exists. Therefore, it can be concluded that f is continuous. The function, however, is not differentiable because:
The values of $f'(0) = \lim_{x\to 0} \frac{f(x)-f(0)}{x-0}$ are different.

The remaining students (4 out of 39), for example, Feri, provided straightforward explanations for why f is continuous but not differentiable at x = 0. They used a counterexample to demonstrate the relationship between continuity and differentiability; that is, there is a condition in which f is continuous at x = a but is not differentiable at the point. He defined the continuity condition when x approaches 0 from the left and right; the value of the function (f(0)) will be the same, indicating that f is continuous at x = 0. Regarding differentiability, he was familiar with the definitions of derivatives used to represent the gradient of a curve's tangent. Thus, he contended that the slope of f is different on the left and right and that the function is not differentiable at x = 0. We classify the meanings he constructs as covariational because he interprets the curve of f using the relationship between two quantities. Our findings should guide students as they attempt to grasp the relationship between continuity and differentiability, as stated in the following interview.

Interviewer : What do you think about your answer saying that f is continuous but is not differentiable at $x = 0$?

Student 3 : ⋯ [pause for a moment]⋯ When x approaches 0, the function's values are the same from the left and the right. Next, the function value is the same. Thus, f is continuous. Then, its gradient increases, increasing from the left of 0. From the left, the gradient also increases, but their values are different. Then, the curve f is not differentiable at $x = 0$,

According to the findings, students developed three meanings when solving graphic problems involving the relationship between continuity and differentiability. There are three types of meanings: physical, analytical, and covariational.

The findings in this section concern the relationship between continuity and differentiability that students construct in the graphic problem. The first was physical meaning, in which students used a particular graph behaviour to solve problems. Some participants developed analytical or covariational meanings. Learners developing the former required symbols or signs to translate the problems' situations. In the latter case, students concurrently constructed meaning by utilizing two related quantities. We would discuss the definitions in greater detail and their implications for practice and calculus curriculum in the classroom.

Students strove to explore the function formula in the graphic context to conclude the continuity and differentiability of a point. They did not, however, understand the implied meanings of the function's graph, such as the rate of change of the value of x toward y, the increase and decrease of the f curve, its gradient, or a particular condition of the curve. According to our findings, most students failed to interpret the propositions of the continuity and differentiability theorems (156 out of 195). They reasoned that if f is differentiable at c, it must also be continuous at c. However, the participants were unaware that the proposition's converse was incorrect. These findings align with Viholainen (2008), who stated that some students believed that the continuity condition resulted in the requirement for differentiability being met. Students' difficulties with the proposition can be minimized by encouraging them to think reversibly about the proposition statements, to use counterexamples, to control their mental schemas, and to create a visual schema (concept map) based on the relationships between the concepts presented (Ikram, Purwanto, & Parta, 2020; & Sevimli, 2018b).

When students identified the continuity and differentiability in the graphic problems, most participants observed the behaviour and the shape of the graph of f. For instance, in Task 1, 8 out of 195 students interpreted the increase or decrease of the curve f around $x = 0$, and predicted the function formula of the curve f as a piecewise-defined function. Based on their answers, they realized that the continuity of f at $x = 0$ was caused by its discontinuous curve (in the form of jump, hole, or vertical asymptote). In terms of differentiability, they thought that the f curve had a sharp corner at $x = 0$ and was similar to the function $f(x) = |x|$, which is not differentiable at $x = 0$. It implied that participants used physical meaning to answer the questions. The situation is called a graph of f has a sharp corner, which means that its left-sided and right-sided derivatives are different (Sevimli, 2018b). When the problems were extended, they were also aware that if the graph of f has a vertical tangent, then it is continuous, but is not differentiable at the point. Students' flow of thinking which use physical meaning tended to carry out a decomposition of the problem or solve it part by part (for example, students solve the continuity and then continue to its differentiability). They separated the problems into sub-section based on the sequence of the problems and proceeded by analyzing each sub-section separately (Rich, Yadav, & Schwarz, 2019).

Another finding showed that some students construct the relation between continuity and differentiability using limit as a basis. For example, if f is continuous at $x = 0$, it is translated to $\lim_{x \to 0} f(x)$ exists, and f is differentiable at $x = 0$ translated to $\lim_{x \to 0} \frac{f(x)-f(0)}{x-0}$ is defined.

It shows thinking of symbols or notations influences the students to solve problems, and their mental schemas activate the idea about the definitions of continuity and derivative (García-García & Dolores-Flores, 2019). In the case of differentiability, our findings are consistent with a study by Park (2015) showing that students were aware of the meanings of differentiability $f'(0)$ represented as

$$\frac{f(x)-f(0)}{x-0} \text{ and add the symbol of } \lim_{x\to 0} \text{ in } \frac{f(x)-f(0)}{x-0}$$

Represented as the tangent in the graph at x = 0. Our results also supported the study by Sánchez-Matamoros, Fernández, & Llinares (2014), revealing that the limit of the difference quotient to the meanings of derivative as tangent are essential parts of students' analysis in identifying and interpreting the elements of the problem. Moreover, students' thinking tends to be dominated by symbol sense, causing every information in the task to be expressed symbolically (Zehavi, 2004). It shows that using analytical meaning by utilizing symbols or notation become the important part of this research findings.

The next findings are related to students' covariational perspectives, in which they viewed problems through the lens of proportion, rate of change, and the function's two variables. Learners described the relationship between two changing quantities, such as: (1) when x approaches zero from the left and right, the value of the function approaches zero as well; (2) the gradients of the f curve from the left and right are different, indicating that f is not differentiable at x = 0. In this instance, students engaged in cognitive activities by simultaneously coordinating changes and variations in quantity (Carlson, Madison, & West, 2015; Sand, Lockwood, & Caballero, 2022; Scheiner, Godino, & Montes, 2022). Additionally, Sevimli's (2018b) observation of the relationship between continuity and differentiability was similar to the covariational meaning. Students occasionally expressed the relationship between quantities when interpreting the graph in the study. They did not, however, reach a general conclusion. According to Kertil, Erbas, & Cetinkaya (2019), students' ability to synthesize two quantities demonstrates a high level of thinking, and students who use covariational thinking in their problem-solving are uncommon.

Students' prior experiences affect their three meanings of the relationship between continuity and differentiability. They create new knowledge based on prior knowledge to solve problems (Martin & Towers, 2016). Learners who correctly answered could recall their prior knowledge, whereas those with incorrect solutions required guidance to understand the mathematical relationship between the problems. Additionally, the role of teachers in the learning process is a factor that influences the use of prior knowledge (Hong & Thomas, 2015). Thus, the findings strongly relate to integrating students' experiences and classroom lessons, particularly regarding continuity and differentiability problems.

The Implication of the Relationship Between Continuity and Differentiability in the Classroom

A cursory review of the calculus textbooks used in Indonesia was conducted to ascertain students' understanding of the importance of continuity and sociability. We were aware that most students encountered procedural problems more frequently than graphical and analytical ones, and we encouraged students to seek out a counterexample. The authors of the textbooks only used how the differentiability of f(x) = |x| at x = 0 is, which is presented analytically and graphically. Nonetheless, there was no explicit emphasis in its practice on the importance of the relationship between continuity and differentiability. Furthermore, none of the authors examined the impact of textbooks on students' perspectives on unfamiliar problems. As a result, we were unsurprised that most learners could not distinguish between when the function f is differentiable and the symbols in the graph of f that cause it to be differentiable.

Three meanings developed by students should be used to help them gain a complete understanding. Learners' conceptual and procedural comprehension may improve (Scheibling-Sève, Pasquinelli, & Sander, 2020). Additionally, the findings indicated that the three meanings students constructed should be considered as they work to develop a more productive meaning of continuity and differentiability. The difficulties students encountered in comprehending the two concepts should be investigated further. As educators, we expect students to (1) learn how to determine whether a function is continuous or differentiable using various representations; (2) understand the situations and conditions under which the f curve is continuous but not differentiable; (3) apply continuity and differentiability theorems that require knowledge of propositions; and (4) explicitly pay more attention to physical, analytic, and symbolic meanings. To help students develop a strong conceptual understanding of the two concepts, we suggest that teachers provide numerous opportunities for students to construct their ideas productively in calculus rather than focusing exclusively on specific problems.

CONCLUSION

This research aimed to determine the meanings that students construct when they apply the relationship between continuity and differentiability to graph problems. Students developed three types of meanings through thematic analysis: physical, analytical, and covariational. The majority of the meanings constructed by students were influenced by the textbooks they used. The three meanings can be used to summarize students' reflections on their thinking while completing the graphic task involving continuity and differentiability. In general, the findings indicated that students' perspectives on problems and efforts to develop their ideas might aid them in analyzing the graphical problem, such as how their beliefs about the continuity and differentiability of a function are represented visually. Due to the graphic problem's centrality in the calculus curriculum, students may underestimate the value of studying the problem in alternative contexts as a new challenge. On the contrary, providing students with explicit classroom representations may be necessary.

The findings can be a foundational theoretical framework for future research on continuity and differentiability in calculus, particularly in learning. This research supports the findings of previous research regarding the critical connection between continuity and differentiability from multiple perspectives. Explicitly, the meanings that students construct should be developed through problem-solving with various representations. These findings are contextualized because most students did not understand the relationship between continuity and differentiability. As a suggestion for future research, the difficulties could be minimized by incorporating the three meanings discovered in future learning.

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